

Quasi-periodic solutions of the Heisenberg hierarchy

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Abstract

The Heisenberg hierarchy and its Hamiltonian structure are derived respectively by virtue of the zero curvature equation and the trace identity. With the help of the Lax matrix we introduce an algebraic curve \mathcal{K}_n of arithmetic genus n , from which we define meromorphic function ϕ and straighten out all of the flows associated with the Heisenberg hierarchy under the Abel-Jacobi coordinates. Finally, we achieve the explicit theta function representations of solutions for the whole Heisenberg hierarchy as a result of the asymptotic properties of ϕ .

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1 Introduction

It is well known that seeking quasi-periodic solutions of soliton equations is a very important topic in soliton theory because soliton equations describe various nonlinear phenomena in natural and applied sciences such as fluid dynamics, plasma physics, solid state

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physics, optical fibers, acoustics, mechanics, biology and mathematical finance. With the development of soliton theory, many approaches were developed from which quasi-periodic solutions for several soliton equations have been obtained in Refs. [1-21], such as the KdV, mKdV, nonlinear Schrödinger, sine-Gordon, Toda lattice and Camassa-Holm equations, etc.

In this paper, we would like to construct quasi-periodic solutions of the Heisenberg hierarchy by means of the methods in Refs. [15,21]. Ferromagnetic chain equation was first proposed in 1935 by Landau and Lifshitz when studying the dispersive theory for magnetic conductivity in magnetic materials [22]. It is an important dynamical equation, which has coherent and chaotic structures depending on the nature of magnetic interactions, and frequently appears in condensation physics, quantum physics and other physics fields [23-25]. The Lax integration of the continuous Heisenberg spin chain equation are studied by Takhtajan through the inverse scattering transform method in 1977 [26]. Almost at the same time the single-soliton solution of Heisenberg spin chain in the isotropic case are obtained by Tjon and Wright [27]. The classical solutions of the continuous Heisenberg spin chain has been obtained by Jevicki and Papanicolaou using a path integral formalism that allows for semi-classical quantization of systems with spin degrees of freedom in 1979 [28]. Soon after, an explicit expression is obtained for the Miura transformation which maps the solutions of the continuous Anisotropic Heisenberg Spin Chain on solutions of the Nonlinear Schrödinger equation by Quispel and Capel in 1983 [29]. Li and Chen gave the higher order Heisenberg spin chain equations, and they proved that these evolution equations are equivalent to the evolution equation of AKNS type in 1986 [30]. Afterwards, The role of nonlocal conservation laws and the corresponding charges are analyzed in the supersymmetric Heisenberg spin chain in 1994 [31]. Cao discussed the parametric representation of the finite-band solution of the Heisenberg equation [32]. The algebraic Bethe ansatz equation has been set up for an open Heisenberg spin chain having an impurity of a different type of spin [33]. Qiao gave the involutive solutions of the higher order Heisenberg spin chain equations by virtue of the spectral problem nonlinearization method [34]. Recently, Du derived the Poisson reduction and Lie-Poisson structure for the nonlinearized spectral problem of the Heisenberg hierarchy by the method of invariants [35]. Wang and Zanardi showed that in one-dimensional isotropic Heisenberg model two-

qubit thermal entanglement and maximal violation of Bell inequalities are directly related with a thermodynamical state function [36]. Wang studied the Darboux transformation for the Heisenberg hierarchy and constructed explicit soliton solutions for the hierarchy by using the Darboux transformation [37]. Guo and others proved the existence of periodic weak solutions to the classical one-dimensional isotropic biquadratic Heisenberg spin chain in 2007 [38]. Its and Korepin considered the XY quantum spin chain in a transverse magnetic field in 2010 [39]. Li and others investigated the gauge transformation between the first-order nonisospectral and isospectral Heisenberg hierarchies [40]. Miszczak and others studied a quantum version of a penny flip game in the Heisenberg model [41].

The paper is structured as follows. In section 2, with the aid of the zero-curvature equation and the trace identity we derive the Heisenberg hierarchy and its Hamiltonian structure. In section 3, the nonlinear recursion relations of the homogeneous coefficients are given based on a Lax matrix and an algebraic curve \mathcal{K}_n of arithmetic genus n . In section 4, we first get the Dubrovin-type equations of the elliptic variables, then we straightened out all the flows of the Heisenberg hierarchy under the Abel-Jacobi coordinates. In section 5, we constructed the quasi-periodic solutions of the whole Heisenberg hierarchy by use of the Riemann theta functions according to the asymptotic properties of the meromorphic function ϕ .

2 The Heisenberg hierarchy and its Hamiltonian structure

In this section, we shall derive the Heisenberg hierarchy associated with the 2×2 spectral problem [26]

$$\varphi_x = U\varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad U = \lambda \begin{pmatrix} w & u \\ v & -w \end{pmatrix}, \quad (2.1)$$

where $w^2 + uv = 1$, u and v are two potentials, λ is a constant spectral parameter. To this end, we solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = \lambda \begin{pmatrix} -\frac{1}{2}wA & B - \frac{1}{2}uA \\ C - \frac{1}{2}vA & \frac{1}{2}wA \end{pmatrix}, \quad (2.2)$$

which is equivalent to

$$\begin{aligned}\frac{1}{2}(wA)_x + \lambda(uC - vB) &= 0, \\ B_x - \frac{1}{2}(uA)_x - 2\lambda wB &= 0, \\ C_x - \frac{1}{2}(vA)_x + 2\lambda wC &= 0,\end{aligned}\tag{2.3}$$

where

$$A = \sum_{j \geq 0} a_{j-1} \lambda^{-j}, \quad B = \sum_{j \geq 0} b_{j-1} \lambda^{-j}, \quad C = \sum_{j \geq 0} c_{j-1} \lambda^{-j}.\tag{2.4}$$

A direct calculation shows that (2.3) and (2.4) imply the Lenard recursion equations

$$KL_{j-1} = JL_j, \quad JL_{-1} = 0, \quad L_j = (c_j, b_j, a_j)^T\tag{2.5}$$

in which K and J are two operators defined by

$$K = \begin{pmatrix} 0 & \partial & -\frac{1}{2}\partial u \\ \partial & 0 & -\frac{1}{2}\partial v \\ u\partial & v\partial & -\partial \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 2w & 0 \\ -2w & 0 & 0 \\ u\partial & v\partial & -\partial \end{pmatrix}.\tag{2.6}$$

We here take

$$L_{-1} = (0, 0, -2)^T\tag{2.7}$$

as a starting point. It is easy to see that $\text{Ker} J = \{\bar{c}_0 L_{-1} \mid \bar{c}_0 \in \mathbb{R}\}$. Then L_j is uniquely determined by the recursion relation (2.5) up to a term $\text{const.} L_{-1}$, which is always assumed to be zero. The first two members are

$$L_0 = \frac{1}{2w} \begin{pmatrix} -v_x \\ u_x \\ u_x v - uv_x \end{pmatrix}, \quad L_1 = \frac{1}{4w} \begin{pmatrix} v_{xx}w - vw_{xx} \\ u_{xx}w - uw_{xx} \\ -2w_{xx} - 3w(u_x v_x + w_x^2) \end{pmatrix}.\tag{2.8}$$

Assume that the time evolution of the eigenfunction φ obeys the differential equation

$$\varphi_{t_m} = V^{(m)} \varphi, \quad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix},\tag{2.9}$$

where $V_{11}^{(m)}, V_{12}^{(m)}, V_{21}^{(m)}$ are polynomials of the spectral parameter λ with

$$\begin{aligned} V_{11}^{(m)} &= \sum_{j=0}^m \left(-\frac{1}{2} w a_{j-1} \right) \lambda^{m+1-j}, \\ V_{12}^{(m)} &= \sum_{j=0}^m \left(b_{j-1} - \frac{1}{2} u a_{j-1} \right) \lambda^{m+1-j}, \\ V_{21}^{(m)} &= \sum_{j=0}^m \left(c_{j-1} - \frac{1}{2} v a_{j-1} \right) \lambda^{m+1-j}. \end{aligned} \quad (2.10)$$

Then the compatibility condition of (2.1) and (2.9) yields the zero curvature equation, $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$u_{t_m} = 2w b_m, \quad v_{t_m} = -2w c_m. \quad (2.11)$$

The first two nontrivial members in the hierarchy are as follows

$$u_{t_1} = \frac{1}{2}(u_{xx}w - uw_{xx}), \quad v_{t_1} = \frac{1}{2}(w_{xx}v - wv_{xx}), \quad (2.12)$$

$$u_{t_2} = \frac{1}{4}u_{xxx} + \frac{3}{8}(uu_xv_x + uw_x^2)_x, \quad v_{t_2} = \frac{1}{4}v_{xxx} + \frac{3}{8}(vu_xv_x + vw_x^2)_x. \quad (2.13)$$

In the following we derive the Hamiltonian structure of the hierarchy (2.11). A direct calculation gives

$$\text{tr} \left(V \frac{\partial U}{\partial \lambda} \right) = \lambda(uC + vB - A), \quad \text{tr} \left(V \frac{\partial U}{\partial u} \right) = \lambda^2 C, \quad \text{tr} \left(V \frac{\partial U}{\partial v} \right) = \lambda^2 B. \quad (2.14)$$

Substituting (2.14) into the trace identity [42] leads to

$$\left(\begin{array}{c} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \end{array} \right) (\lambda(uC + vB - A)) = \lambda^{-\gamma} \left(\frac{\partial}{\partial \lambda} \lambda^\gamma \left(\begin{array}{c} \lambda^2 C \\ \lambda^2 B \end{array} \right) \right). \quad (2.15)$$

Comparing the coefficient of the λ^{-n} in (2.15) yields

$$\left(\begin{array}{c} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \end{array} \right) (uc_n + vb_n - a_n) = (\gamma - n + 1) \left(\begin{array}{c} c_n \\ b_n \end{array} \right). \quad (2.16)$$

Let $n = 1$ in (2.16) and find that $\gamma = -1$, so we have

$$\frac{\delta H_n}{\delta \eta} = \left(\begin{array}{c} c_n \\ b_n \end{array} \right), \quad \eta = \left(\begin{array}{c} u \\ v \end{array} \right), \quad H_n = \frac{a_n - uc_n - vb_n}{n}. \quad (2.17)$$

Therefore, the Hamiltonian structure of the Heisenberg hierarchy (2.11) is as follows

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = \bar{J} \frac{\delta H_n}{\delta \eta}, \quad \bar{J} = \begin{pmatrix} 0 & 2w \\ -2w & 0 \end{pmatrix}. \quad (2.18)$$

3 Nonlinear recursion relations

Let $\chi = (\chi_1, \chi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ be two basic solutions of (2.1) and (2.9). We introduce a Lax matrix

$$W = \frac{1}{2}(\chi\psi^T + \psi\chi^T) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \lambda \begin{pmatrix} G & F \\ H & -G \end{pmatrix} \quad (3.1)$$

which satisfies the Lax equations

$$W_x = [U, W], \quad W_{t_m} = [V^{(m)}, W]. \quad (3.2)$$

Therefore, $\det W$ is a constant independent of x and t_m . Equation (3.2) can be written as

$$\begin{aligned} G_x &= \lambda(uH - vF), \\ F_x &= 2\lambda(wF - uG), \\ H_x &= 2\lambda(vG - wH), \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} G_{t_m} &= V_{12}^{(m)}H - V_{21}^{(m)}F, \\ F_{t_m} &= 2(V_{11}^{(m)}F - V_{12}^{(m)}G), \\ H_{t_m} &= 2(V_{21}^{(m)}G - V_{11}^{(m)}H). \end{aligned} \quad (3.4)$$

Suppose functions F , G and H are finite-order polynomials in λ

$$G = \sum_{j=0}^{n+1} G_{j-1} \lambda^{n+1-j}, \quad F = \sum_{j=0}^{n+1} F_{j-1} \lambda^{n+1-j}, \quad H = \sum_{j=0}^{n+1} H_{j-1} \lambda^{n+1-j}, \quad (3.5)$$

where

$$G_{j-1} = -\frac{1}{2}wg_{j-1}, \quad F_{j-1} = f_{j-1} - \frac{1}{2}ug_{j-1}, \quad H_{j-1} = h_{j-1} - \frac{1}{2}vg_{j-1}. \quad (3.6)$$

Substituting (3.5) and (3.6) into (3.3) yields

$$KE_{j-1} = JE_j, \quad JE_{-1} = 0, \quad (3.7)$$

$$KE_n = 0, \quad (3.8)$$

where $E_j = (h_j, f_j, g_j)^T$, $-1 \leq j \leq n$. It is easy to see that the equation $JE_{-1} = 0$ has the general solution

$$E_{-1} = \alpha_{-1}(0, 0, -2)^T. \quad (3.9)$$

Without loss of generality, let $\alpha_{-1} = 1$. If we take (3.9) as a starting point, then E_j can be recursively determined by the relation (3.7). Acting with the operator $(J^{-1}K)^{k+1}$ upon (3.9), we obtain from (3.7) and (2.5) that

$$E_k = \sum_{j=0}^{k+1} \alpha_{j-1} L_{k-j}, \quad -1 \leq k \leq n, \quad (3.10)$$

where $\alpha_0, \alpha_1, \dots, \alpha_k$ are constants of integration. The first two members in (3.10) are

$$E_0 = \begin{pmatrix} -\frac{v_x}{2w} \\ \frac{u_x}{2w} \\ \frac{u_x v - uv_x}{2w} - 2\alpha_0 \end{pmatrix}, \quad (3.11)$$

$$E_1 = \begin{pmatrix} \frac{v_{xx}w - vw_{xx}}{4w} - \alpha_0 \frac{v_x}{2w} \\ \frac{u_{xx}w - uw_{xx}}{4w} + \alpha_0 \frac{u_x}{2w} \\ \frac{-2w_{xx} - 3w(u_x v_x + w_x^2)}{4w} + \alpha_0 \frac{u_x v - uv_x}{2w} - 2\alpha_1 \end{pmatrix}. \quad (3.12)$$

Since $\det W$ is a $(2n+4)$ th-order polynomial in λ , whose coefficients are constants independent of x and t_m , we have

$$-\det W = \lambda^2(G^2 + FH) = \lambda^2 \prod_{j=1}^{2n+2} (\lambda - \lambda_j) = \lambda^2 R(\lambda), \quad (3.13)$$

one is naturally led to introduce the hyperelliptic curve \mathcal{K}_n of arithmetic genus n defined by

$$\mathcal{K}_n : y^2 - R(\lambda) = 0. \quad (3.14)$$

The curve \mathcal{K}_n can be compactified by joining two points at infinity, $P_{\infty\pm}$, where $P_{\infty+} \neq P_{\infty-}$. For notational simplicity the compactification of the curve \mathcal{K}_n is also denoted by \mathcal{K}_n . Here we assume that the zeros λ_j of $R(\lambda)$ in (3.13) are mutually distinct, which means $\lambda_j \neq \lambda_k$, for $j \neq k$, $1 \leq j, k \leq 2n+2$. Then the hyperelliptic curve \mathcal{K}_n becomes nonsingular.

From the following lemma, we can explicitly represent $\alpha_l(-1 \leq l \leq n)$ by the constants $\lambda_1, \dots, \lambda_{2n+2}$.

Lemma 3.1.

$$\alpha_l = c_l(\underline{\Lambda}), \quad l = -1, \dots, n, \quad (3.15)$$

where

$$\begin{aligned} \underline{\Lambda} &= (\lambda_1, \dots, \lambda_{2n+2}), \quad c_{-1}(\underline{\Lambda}) = 1, \quad c_0(\underline{\Lambda}) = -\frac{1}{2} \sum_{j=1}^{2n+2} \lambda_j, \dots, \\ c_l(\underline{\Lambda}) &= \sum_{\substack{j_1, \dots, j_{2n+2}=0 \\ j_1 + \dots + j_{2n+2} = l+1}}^{l+1} \frac{(2j_1)! \dots (2j_{2n+2})! \lambda_1^{j_1} \dots \lambda_{2n+2}^{j_{2n+2}}}{2^{2l+2} (j_1!)^2 \dots (j_{2n+2}!)^2 (2j_1 - 1) \dots (2j_{2n+2} - 1)}. \end{aligned} \quad (3.16)$$

Proof. Let

$$\hat{F}_j = F_j|_{\alpha_0=\dots=\alpha_j=0}, \quad \hat{H}_j = H_j|_{\alpha_0=\dots=\alpha_j=0}, \quad \hat{G}_j = G_j|_{\alpha_0=\dots=\alpha_j=0}. \quad (3.17)$$

It will be convenient to introduce the notion of a degree, $\deg(\cdot)$, to effectively distinguish between homogeneous and nonhomogeneous quantities. Define

$$\deg(u) = 0, \quad \deg(v) = 0, \quad \deg(w) = 0, \quad \deg(\partial_x) = 1, \quad (3.18)$$

thus from (3.7) and (3.10) it can be implied that

$$\deg(\hat{F}_k) = k + 1, \quad \deg(\hat{H}_k) = k + 1, \quad \deg(\hat{G}_k) = k + 1, \quad k \in \mathbb{N}_0 \cup \{-1\}. \quad (3.19)$$

Temporarily fixed the branch of $R(\lambda)^{1/2}$ as λ^{n+1} near infinity, $R(\lambda)^{-1/2}$ has the following expansion

$$R(\lambda)^{-1/2} \underset{\lambda \rightarrow \infty}{=} \sum_{l=0}^{\infty} \hat{c}_{l-1}(\underline{\Lambda}) \lambda^{-n-1-l}, \quad (3.20)$$

where

$$\begin{aligned} \underline{\Lambda} &= (\lambda_1, \dots, \lambda_{2n+2}), \quad \hat{c}_{-1}(\underline{\Lambda}) = 1, \quad \hat{c}_0(\underline{\Lambda}) = \frac{1}{2} \sum_{j=1}^{2n+2} \lambda_j, \dots, \\ \hat{c}_l(\underline{\Lambda}) &= \sum_{\substack{j_1, \dots, j_{2n+2}=0 \\ j_1 + \dots + j_{2n+2} = l+1}}^{l+1} \frac{(2j_1)! \dots (2j_{2n+2})! \lambda_1^{j_1} \dots \lambda_{2n+2}^{j_{2n+2}}}{2^{2l+2} (j_1!)^2 \dots (j_{2n+2}!)^2}. \end{aligned} \quad (3.21)$$

Dividing $F(\lambda)$, $H(\lambda)$, $G(\lambda)$ by $R(\lambda)^{1/2}$ near infinity respectively, we obtain

$$\begin{aligned}\frac{F(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda \rightarrow \infty}{=} \left(\sum_{l=0}^{n+1} F_{l-1} \lambda^{n+1-l} \right) \left(\sum_{l=0}^{\infty} \hat{c}_{l-1}(\underline{\Lambda}) \lambda^{-n-1-l} \right) = \sum_{l=0}^{\infty} \check{F}_{l-1} \lambda^{-l}, \\ \frac{H(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda \rightarrow \infty}{=} \left(\sum_{l=0}^{n+1} H_{l-1} \lambda^{n+1-l} \right) \left(\sum_{l=0}^{\infty} \hat{c}_{l-1}(\underline{\Lambda}) \lambda^{-n-1-l} \right) = \sum_{l=0}^{\infty} \check{H}_{l-1} \lambda^{-l}, \\ \frac{G(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda \rightarrow \infty}{=} \left(\sum_{l=0}^{n+1} G_{l-1} \lambda^{n+1-l} \right) \left(\sum_{l=0}^{\infty} \hat{c}_{l-1}(\underline{\Lambda}) \lambda^{-n-1-l} \right) = \sum_{l=0}^{\infty} \check{G}_{l-1} \lambda^{-l},\end{aligned}\tag{3.22}$$

for some coefficients \check{F}_{l-1} , \check{H}_{l-1} , \check{G}_{l-1} to be determined next. Noticing (3.3) and (3.13), we get

$$\begin{aligned}-\frac{F_{xx}F}{2\lambda^2 u^2} + \frac{F_x^2}{4\lambda^2 u^2} + \frac{u_x}{2\lambda^2 u^3} F F_x + F^2 \left(\frac{1}{u^2} + \frac{w_x u - w u_x}{\lambda u^3} \right) &= R(\lambda), \\ -\frac{H_{xx}H}{2\lambda^2 v^2} + \frac{H_x^2}{4\lambda^2 v^2} + \frac{v_x}{2\lambda^2 v^3} H H_x + H^2 \left(\frac{1}{v^2} + \frac{w v_x - w_x v}{\lambda v^3} \right) &= R(\lambda),\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}-\frac{1}{2\lambda u} G F_x + \frac{w}{u} G F + F H &= R(\lambda), \\ \frac{1}{2\lambda v} G H_x + \frac{w}{v} G H + F H &= R(\lambda), \\ \frac{1}{\lambda u} F G_x + \frac{v}{u} F^2 + G^2 &= R(\lambda).\end{aligned}\tag{3.24}$$

Respectively substituting (3.22) into (3.23), (3.13) and (3.24), comparing the coefficients of λ with the same power, we arrive at the following recursive relations

$$\begin{aligned}\check{F}_{k-1} &= -\frac{1}{2u} \left\{ \sum_{l=0}^{k-2} \left(-\frac{1}{2} \check{F}_{l-1,xx} \check{F}_{k-3-l} + \frac{1}{4} \check{F}_{l-1,x} \check{F}_{k-3-l,x} + \frac{u_x}{2u} \check{F}_{l-1,x} \check{F}_{k-3-l} \right) \right. \\ &\quad \left. + \sum_{l=1}^{k-1} \check{F}_{l-1} \check{F}_{k-1-l} + \sum_{l=0}^{k-1} \frac{w_x u - w u_x}{u} \check{F}_{l-1} \check{F}_{k-2-l} \right\},\end{aligned}\tag{3.25}$$

$$\begin{aligned}\check{H}_{k-1} &= -\frac{1}{2v} \left\{ \sum_{l=0}^{k-2} \left(-\frac{1}{2} \check{H}_{l-1,xx} \check{H}_{k-3-l} + \frac{1}{4} \check{H}_{l-1,x} \check{H}_{k-3-l,x} + \frac{v_x}{2v} \check{H}_{l-1,x} \check{H}_{k-3-l} \right) \right. \\ &\quad \left. + \sum_{l=1}^{k-1} \check{H}_{l-1} \check{H}_{k-1-l} + \sum_{l=0}^{k-1} \frac{w v_x - w_x v}{v} \check{H}_{l-1} \check{H}_{k-2-l} \right\},\end{aligned}\tag{3.26}$$

$$\check{G}_{k-1} = -\frac{1}{2w} \sum_{l=0}^k \check{F}_{l-1} \check{H}_{k-1-l} - \frac{1}{2w} \sum_{l=1}^{k-1} \check{G}_{l-1} \check{G}_{k-1-l}\tag{3.27}$$

for $k \geq 2$, and relations

$$\check{F}_{k-1,x} = -\frac{1}{w} \sum_{l=0}^{k-1} \check{F}_{l-1,x} \check{G}_{k-1-l} + \sum_{l=0}^{k+1} \left(2\check{F}_{l-1} \check{G}_{k-l} + \frac{2u}{w} \check{F}_{l-1} \check{H}_{k-l} \right), \quad (3.28)$$

$$\check{H}_{k-1,x} = -\frac{1}{w} \sum_{l=0}^{k-1} \check{H}_{l-1,x} \check{G}_{k-1-l} - \sum_{l=0}^{k+1} \left(2\check{H}_{l-1} \check{G}_{k-l} + \frac{2v}{w} \check{F}_{l-1} \check{H}_{k-l} \right), \quad (3.29)$$

$$\check{G}_{k-1,x} = -\frac{1}{u} \sum_{l=0}^{k-1} \check{G}_{l-1,x} \check{F}_{k-1-l} - \sum_{l=0}^{k+1} \left(\check{G}_{l-1} \check{G}_{k-l} + \frac{v}{u} \check{F}_{l-1} \check{F}_{k-l} \right) \quad (3.30)$$

for $k \geq 1$ and

$$\begin{aligned} \check{F}_{-1} &= u, & \check{F}_0 &= \frac{1}{2}(wu_x - w_x u), \\ \check{H}_{-1} &= v, & \check{H}_0 &= \frac{1}{2}(w_x v - wv_x), \\ \check{G}_{-1} &= w, & \check{G}_0 &= \frac{1}{4}(uv_x - u_x v). \end{aligned} \quad (3.31)$$

The signs of \check{F}_{-1} , \check{H}_{-1} and \check{G}_{-1} have been chosen such that $\check{F}_{-1} = \hat{F}_{-1}$, $\check{H}_{-1} = \hat{H}_{-1}$ and $\check{G}_{-1} = \hat{G}_{-1}$. Moreover, we can prove inductively using the nonlinear recursion relations (3.25)-(3.27) and (3.31) that

$$\deg(\check{F}_k) = k+1, \quad \deg(\check{H}_k) = k+1, \quad \deg(\check{G}_k) = k+1, \quad k \in \mathbb{N}_0 \cup \{-1\}. \quad (3.32)$$

It can be proved inductively that

$$\begin{aligned} \check{F}_{k-1,x} + 2u\check{G}_k &= 2w\check{F}_k, \\ \check{H}_{k-1,x} - 2v\check{G}_k &= -2w\check{H}_k, \\ \check{G}_{k-1,x} &= u\check{H}_k - v\check{F}_k, \end{aligned} \quad (3.33)$$

for $k \in \mathbb{N}_0 \cup \{-1\}$. In fact, suppose that for arbitrary l , $-1 \leq l \leq k-2$, we have

$$\begin{aligned} \check{F}_{l,x} + 2u\check{G}_{l+1} &= 2w\check{F}_{l+1}, \\ \check{H}_{l,x} - 2v\check{G}_{l+1} &= -2w\check{H}_{l+1}, \\ \check{G}_{l,x} &= u\check{H}_{l+1} - v\check{F}_{l+1}, \end{aligned} \quad (3.34)$$

then with the help of (3.27), (3.28) and (3.31), it can be calculated out that

$$\begin{aligned}
& \check{F}_{k-1,x} + 2u\check{G}_k \\
&= -\frac{1}{w} \sum_{l=0}^{k-1} \check{F}_{l-1,x} \check{G}_{k-1-l} + \sum_{l=0}^{k+1} \left(2\check{F}_{l-1} \check{G}_{k-l} + \frac{2u}{w} \check{F}_{l-1} \check{H}_{k-l} \right) + 2u\check{G}_k \\
&= -\frac{1}{w} \sum_{l=0}^{k-1} (2w\check{F}_l - 2u\check{G}_l) \check{G}_{k-1-l} + \sum_{l=0}^{k+1} \left(2\check{F}_{l-1} \check{G}_{k-l} + \frac{2u}{w} \check{F}_{l-1} \check{H}_{k-l} \right) + 2u\check{G}_k \\
&= 2 \left(\sum_{l=0}^{k+1} \check{F}_{l-1} \check{G}_{k-l} - \sum_{l=0}^{k-1} \check{F}_l \check{G}_{k-1-l} \right) + \frac{2u}{w} \left(\sum_{l=1}^k \check{G}_{l-1} \check{G}_{k-l} + \sum_{l=0}^{k+1} \check{F}_{l-1} \check{H}_{k-l} \right) + 2u\check{G}_k \\
&= 2(\check{G}_{-1} \check{F}_k + \check{F}_{-1} \check{G}_k) - 4u\check{G}_k + 2u\check{G}_k \\
&= 2w\check{F}_k.
\end{aligned} \tag{3.35}$$

It can be similarly proved that

$$\check{H}_{k-1,x} - 2v\check{G}_k = -2w\check{H}_k. \tag{3.36}$$

Basing on the above calculation, we have that

$$\begin{aligned}
\check{G}_{k-1,x} &= -\frac{1}{u} \sum_{l=0}^{k-1} \check{G}_{l-1,x} \check{F}_{k-1-l} - \sum_{l=0}^{k+1} \left(\check{G}_{l-1} \check{G}_{k-l} + \frac{v}{u} \check{F}_{l-1} \check{F}_{k-l} \right) \\
&= -\frac{1}{u} \sum_{l=0}^{k-1} (u\check{H}_l - v\check{F}_l) \check{F}_{k-1-l} - \sum_{l=0}^{k+1} \left(\check{G}_{l-1} \check{G}_{k-l} + \frac{v}{u} \check{F}_{l-1} \check{F}_{k-l} \right) \\
&= -\sum_{l=0}^{k+1} \check{F}_{l-1} \check{H}_{k-l} - \sum_{l=1}^k \check{G}_{l-1} \check{G}_{k-l} + u\check{H}_k + v\check{F}_k - 2w\check{G}_k - \frac{2v}{u} \check{F}_{-1} \check{F}_k \\
&= u\check{H}_k - v\check{F}_k.
\end{aligned} \tag{3.37}$$

Hence, \check{F}_l , \check{H}_l and \check{G}_l are equal to \hat{F}_l , \hat{H}_l and \hat{G}_l respectively for all $l \in \mathbb{N}_0 \cup \{-1\}$. Thus we proved

$$\frac{F(\lambda)}{R(\lambda)^{1/2}} \underset{\lambda \rightarrow \infty}{=} \sum_{l=0}^{\infty} \hat{F}_{l-1} \lambda^{-l}, \quad \frac{H(\lambda)}{R(\lambda)^{1/2}} \underset{\lambda \rightarrow \infty}{=} \sum_{l=0}^{\infty} \hat{H}_{l-1} \lambda^{-l}, \quad \frac{G(\lambda)}{R(\lambda)^{1/2}} \underset{\lambda \rightarrow \infty}{=} \sum_{l=0}^{\infty} \hat{G}_{l-1} \lambda^{-l}. \tag{3.38}$$

Considering

$$R(\lambda)^{1/2} \underset{\lambda \rightarrow \infty}{=} \sum_{l=0}^{\infty} c_{l-1}(\underline{\Lambda}) \lambda^{n+1-l}, \tag{3.39}$$

a comparison of the coefficients of λ^{-k} in the following equation

$$1 = R(\lambda)^{1/2} \times R(\lambda)^{-1/2} \underset{\lambda \rightarrow \infty}{=} \left(\sum_{l=0}^{\infty} c_{l-1}(\underline{\Lambda}) \lambda^{n+1-l} \right) \left(\sum_{l=0}^{\infty} \hat{c}_{l-1}(\underline{\Lambda}) \lambda^{-n-1-l} \right) \quad (3.40)$$

yields

$$\sum_{l=0}^k c_{k-l-1}(\underline{\Lambda}) \hat{c}_{l-1}(\underline{\Lambda}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \quad (3.41)$$

Therefore, we compute that

$$\sum_{m=0}^{k+1} c_{k-m}(\underline{\Lambda}) \hat{F}_{m-1} = \sum_{m=0}^{k+1} c_{k-m}(\underline{\Lambda}) \sum_{l=0}^m F_{l-1} \hat{c}_{m-l-1}(\underline{\Lambda}) = \sum_{l=0}^{k+1} F_{l-1} \sum_{p=0}^{k+1-l} c_{k-l-p}(\underline{\Lambda}) \hat{c}_{p-1}(\underline{\Lambda}) = F_k, \quad (3.42)$$

where $k = -1, \dots, n$. \square

4 Dubrovin-type equations and straightening out of the flows

In this section, we introduce elliptic variables and Abel-Jacobi coordinates. Then we derive the system of Dubrovin-type differential equations. The straightening out of various flows is exactly given through the Abel-Jacobi coordinates. Noticing (3.5), we write F and H as finite products which take the form

$$F = u \prod_{j=1}^{n+1} (\lambda - \mu_j), \quad H = v \prod_{j=1}^{n+1} (\lambda - \nu_j), \quad (4.1)$$

where $\{\mu_j\}_{j=1}^{n+1}$ and $\{\nu_j\}_{j=1}^{n+1}$ are called elliptic variables. According to the definition of \mathcal{K}_n , we can lift the roots μ_j and ν_j to \mathcal{K}_n by introducing

$$\hat{\mu}_j(x, t_m) = (\mu_j(x, t_m), -G(\mu_j(x, t_m), x, t_m)), \quad (4.2)$$

$$\hat{\nu}_j(x, t_m) = (\nu_j(x, t_m), G(\nu_j(x, t_m), x, t_m)), \quad (4.3)$$

where $j = 1, \dots, n+1$, $(x, t_m) \in \mathbb{R}^2$.

Noticing (3.13), we obtain

$$G|_{\lambda=\mu_k} = \sqrt{R(\mu_k)}, \quad G|_{\lambda=\nu_k} = \sqrt{R(\nu_k)}. \quad (4.4)$$

By virtue of (3.3) and (4.1), we obtain

$$F_x|_{\lambda=\mu_k} = -u\mu_{k,x} \prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\mu_k - \mu_j) = -2u\mu_k G|_{\lambda=\mu_k}, \quad (4.5)$$

$$H_x|_{\lambda=\nu_k} = -v\nu_{k,x} \prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\nu_k - \nu_j) = 2v\nu_k G|_{\lambda=\nu_k}. \quad (4.6)$$

From (4.4)-(4.6) we have

$$\mu_{k,x} = \frac{2\mu_k \sqrt{R(\mu_k)}}{\prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\mu_k - \mu_j)}, \quad \nu_{k,x} = \frac{-2\nu_k \sqrt{R(\nu_k)}}{\prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\nu_k - \nu_j)}, \quad 1 \leq k \leq n+1. \quad (4.7)$$

Similarly, we get the evolution of $\{\mu_j\}$ and $\{\nu_j\}$ along the t_m -flow

$$\mu_{k,t_m} = \frac{2V_{12}^{(m)}(\mu_k) \sqrt{R(\mu_k)}}{u \prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\mu_k - \mu_j)}, \quad \nu_{k,t_m} = \frac{-2V_{21}^{(m)}(\nu_k) \sqrt{R(\nu_k)}}{v \prod_{\substack{j=1 \\ j \neq k}}^{n+1} (\nu_k - \nu_j)}, \quad 1 \leq k \leq n+1. \quad (4.8)$$

In order to straighten out of the corresponding flows, we equip \mathcal{K}_n with canonical basis cycles: $\tilde{a}_1, \dots, \tilde{a}_n; \tilde{b}_1, \dots, \tilde{b}_n$, which are independent and have intersection numbers as follows

$$\tilde{a}_j \circ \tilde{a}_k = 0, \quad \tilde{b}_j \circ \tilde{b}_k = 0, \quad \tilde{a}_j \circ \tilde{b}_k = \delta_{jk}. \quad (4.9)$$

For the present, we will choose our basis as the following set^[8]

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad 1 \leq l \leq n, \quad (4.10)$$

which are n linearly independent homomorphic differentials on \mathcal{K}_n . Then the period matrices A and B can be constructed from

$$A_{kj} = \int_{\tilde{a}_j} \tilde{\omega}_k, \quad B_{kj} = \int_{\tilde{b}_j} \tilde{\omega}_k. \quad (4.11)$$

It is possible to show that matrices A and B are invertible [3]. Now we define the matrices C and τ by $C = A^{-1}$, $\tau = A^{-1}B$. The matrix τ can be shown to be symmetric ($\tau_{kj} = \tau_{jk}$), and it has positive definite imaginary part ($\text{Im}\tau > 0$). If we normalize $\tilde{\omega}_l$ into the new basis ω_j ,

$$\omega_j = \sum_{l=1}^n C_{jl} \tilde{\omega}_l, \quad 1 \leq j \leq n, \quad (4.12)$$

then we obtain

$$\int_{\tilde{a}_k} \omega_j = \sum_{l=1}^n C_{jl} \int_{\tilde{a}_k} \tilde{\omega}_l = \delta_{jk}, \quad \int_{\tilde{b}_k} \omega_j = \tau_{jk}. \quad (4.13)$$

Let \mathcal{T}_n be the period lattice $\mathcal{T}_n = \{\underline{z} \in \mathbb{C}^n | \underline{n} + \tau \underline{m}, \underline{m}, \underline{n} \in \mathbb{Z}^n\}$. The complex torus $\mathcal{T} = \mathbb{C}^n / \mathcal{T}_n$ is called the Jacobian variety of \mathcal{K}_n . Now we introduce the Abel map $\mathcal{A}(P) : \text{Div}(\mathcal{K}_n) \rightarrow \mathcal{T}$

$$\mathcal{A}(P) = \left(\int_{P_0}^P \underline{\omega} \right) (\text{mod } \mathcal{T}_n), \quad \mathcal{A}\left(\sum n_k P_k\right) = \sum n_k \mathcal{A}(P_k), \quad (4.14)$$

where $P, P_k \in \mathcal{K}_n$, $\underline{\omega} = (\omega_1, \dots, \omega_n)$. Considering two special divisors $\sum_{k=1}^{n+1} P_k^{(l)}$, $l = 1, 2$, we define the Abel-Jacobi coordinates as follows

$$\mathcal{A}\left(\sum_{k=1}^{n+1} P_k^{(l)}\right) = \sum_{k=1}^{n+1} \mathcal{A}(P_k^{(l)}) = \sum_{k=1}^{n+1} \int_{P_0}^{P_k^{(l)}} \underline{\omega} = \underline{\rho}^{(l)}, \quad (4.15)$$

with $P_k^{(1)} = \hat{\mu}_k(x, t_m)$, and $P_k^{(2)} = \hat{\nu}_k(x, t_m)$, whose components are

$$\sum_{k=1}^{n+1} \int_{P_0}^{P_k^{(l)}} \omega_j = \rho_j^{(l)}, \quad 1 \leq j \leq n, \quad l = 1, 2. \quad (4.16)$$

Without loss of generality, we choose the branch point $P_0 = (\lambda_{j_0}, 0)$, $j_0 \in \{1, \dots, 2n+2\}$, as a convenient base point, and $\lambda(P_0)$ is its local coordinate. From (4.7), we get

$$\partial_x \rho_j^{(1)} = \sum_{l=1}^n \sum_{k=1}^{n+1} C_{jl} \frac{\mu_k^{l-1} \mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{2C_{jl} \mu_k^l}{\prod_{\substack{r=1 \\ r \neq k}}^{n+1} (\mu_k - \mu_r)} = 2C_{jn}, \quad 1 \leq j \leq n, \quad (4.17)$$

$$\partial_x \rho_j^{(2)} = \sum_{l=1}^n \sum_{k=1}^{n+1} C_{jl} \frac{\nu_k^{l-1} \nu_{k,x}}{\sqrt{R(\nu_k)}} = \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{-2C_{jl} \nu_k^l}{\prod_{\substack{r=1 \\ r \neq k}}^{n+1} (\nu_k - \nu_r)} = -2C_{jn}, \quad 1 \leq j \leq n, \quad (4.18)$$

with the aid of the following equalities:

$$\sum_{k=1}^n \frac{\mu_k^{l-1}}{\prod_{r \neq k} (\mu_k - \mu_r)} = \begin{cases} \delta_{ln}, & 1 \leq l \leq n, \\ \sum_{r_1 + \dots + r_n = l-n, r_j \geq 0} \mu_1^{r_1} \dots \mu_n^{r_n}, & l > n. \end{cases} \quad (4.19)$$

Theorem 4.1. (Straightening out of the t_m -flow)

$$\partial_{t_m} \underline{\rho}^{(1)} = 2 \sum_{l=0}^m \beta_{l-1} \underline{C}_{n-m+l}, \quad (4.20)$$

$$\partial_{t_m} \underline{\rho}^{(2)} = -2 \sum_{l=0}^m \beta_{l-1} \underline{C}_{n-m+l}, \quad (4.21)$$

where $\underline{\rho}^{(i)} = (\rho_1^{(i)}, \dots, \rho_n^{(i)})$, $\underline{C}_k = (C_{1k}, \dots, C_{nk})$, $1 \leq k \leq n$, and the recursive formula:

$$\beta_{-1} = 1, \quad \beta_0 = -\alpha_0, \quad \beta_1 = \alpha_0^2 - \alpha_1, \quad \beta_k = -\sum_{j=0}^k \alpha_j \beta_{k-1-j}. \quad (4.22)$$

Proof. Here we only give the proof of (4.20). Equation (4.21) can be proved in a similar way. Using (3.10), we arrive at

$$F_k = \sum_{j=0}^{k+1} \alpha_{j-1} V_{12,k-j}^{(m)}, \quad (4.23)$$

which implies

$$V_{12,k}^{(m)} = \sum_{j=0}^{k+1} \beta_{j-1} F_{k-j}. \quad (4.24)$$

In fact, it is easy to see that $F_{-1} = V_{12,-1}^{(m)} = u$. Suppose that (4.24) holds. Then a direct calculation shows by (4.23) that

$$\begin{aligned} V_{12,k+1}^{(m)} &= F_{k+1} - \alpha_0 V_{12,k}^{(m)} - \dots - \alpha_k V_{12,0}^{(m)} - \alpha_{k+1} V_{12,-1}^{(m)} \\ &= F_{k+1} - \alpha_0 (\beta_{-1} F_k + \beta_0 F_{k-1} + \dots + \beta_k F_{-1}) - \dots - \alpha_k (\beta_{-1} F_0 + \beta_0 F_{-1}) - \alpha_{k+1} F_{-1} \\ &= F_{k+1} + (-\alpha_0 \beta_{-1}) F_k + (-\alpha_0 \beta_0 - \alpha_1 \beta_{-1}) F_{k-1} + \dots + (-\alpha_0 \beta_k - \dots - \alpha_{k+1}) F_{-1} \\ &= F_{k+1} + \beta_0 F_k + \beta_1 F_{k-1} + \dots + \beta_{k+1} F_{-1} \\ &= \sum_{j=0}^{k+2} \beta_{j-1} F_{k+1-j}. \end{aligned} \quad (4.25)$$

Therefore (4.24) holds. From (4.8), (4.15), (4.19) and (4.24), we have

$$\begin{aligned}
\partial_{t_m} \rho_j^{(1)} &= \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{C_{jl} \mu_k^{l-1} \mu_{k,t_m}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{2C_{jl} \mu_k^{l-1} V_{12}^{(m)}(\mu_k)}{u \prod_{\substack{r=1 \\ r \neq k}}^{n+1} (\mu_k - \mu_r)} \\
&= \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{2C_{jl} \mu_k^{l-1}}{u \prod_{\substack{r=1 \\ r \neq k}}^{n+1} (\mu_k - \mu_r)} \left(\sum_{p=0}^m V_{12,p-1}^{(m)} \mu_k^{m+1-p} \right) \\
&= \sum_{l=1}^n \sum_{k=1}^{n+1} \frac{2C_{jl}}{u \prod_{\substack{r=1 \\ r \neq k}}^{n+1} (\mu_k - \mu_r)} \sum_{p=0}^m \left(\sum_{q=0}^p \beta_{q-1} F_{p-1-q} \right) \mu_k^{m+l-p} \quad (4.26) \\
&= \sum_{q=0}^m \frac{2\beta_{q-1}}{u} \sum_{p=q}^m F_{p-1-q} \sum_{l=0}^{m-p} C_{j,n-(m-p)+l} \Gamma_l \\
&= \sum_{q=0}^m \frac{2\beta_{q-1}}{u} \sum_{k=0}^{m-q} \sum_{l=0}^k C_{j,n-(m-q)+k} F_{l-1} \Gamma_{k-l},
\end{aligned}$$

with

$$\Gamma_0 = 1, \quad \Gamma_k = \sum_{\substack{j_1 + \dots + j_{n+1} = k \\ j_i \geq 0}} \mu_1^{j_1} \dots \mu_{n+1}^{j_{n+1}}, \quad k \geq 1. \quad (4.27)$$

Therefore, we obtain that

$$\partial_{t_m} \rho_j^{(1)} = \sum_{l=0}^m \frac{2\beta_{l-1}}{u} F_{-1} C_{j,n-m+l} = 2 \sum_{l=0}^m \beta_{l-1} C_{j,n-m+l} \quad (4.28)$$

in view of the formula [21]

$$\sum_{\substack{j_1 + j_2 = k \\ j_i \geq 0}} \Gamma_{j_1} F_{j_2-1} = 0, \quad 1 \leq k \leq m, \quad (4.29)$$

where

$$F_{-1} = u, \quad F_0 = -u \sum_{j=1}^{n+1} \mu_j, \quad F_l = (-1)^{l+1} u \sum_{\substack{j_1 < \dots < j_{l+1} \\ j_i \geq 1}} \mu_{j_1} \dots \mu_{j_{l+1}}, \quad 0 \leq l \leq n. \quad (4.30)$$

This completes the proof of the theorem. \square

5 Quasi-periodic solutions

In the section, we shall construct quasi-periodic solutions of the Heisenberg hierarchy (2.11). From (3.13) and (3.14) we have

$$y^2 = G^2 + FH, \quad (5.1)$$

that is

$$(y - G)(y + G) = FH, \quad (5.2)$$

and then we can define the meromorphic function $\phi(P, x, t_m)$ on \mathcal{K}_n

$$\phi(P, x, t_m) = \frac{y - G}{F} = \frac{H}{y + G}, \quad (5.3)$$

where $P = (\lambda, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$.

Lemma 5.1. Suppose that $u(x, t_m), v(x, t_m), w(x, t_m) \in C^\infty(\mathbb{R}^2)$ satisfy the hierarchy (2.11). Let $\lambda_j \in \mathbb{C} \setminus \{0\}$, $1 \leq j \leq 2n + 2$, and $P = (\lambda, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$. Then

$$\phi \underset{\zeta \rightarrow 0}{=} \begin{cases} -\frac{1+w}{u} + \frac{(1+w)u_x - uw_x}{2u^2}\zeta + O(\zeta^2), & \text{as } P \rightarrow P_{\infty+}, \\ \frac{1-w}{u} + \frac{(1-w)u_x + uw_x}{2u^2}\zeta + O(\zeta^2), & \text{as } P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = \lambda^{-1}. \quad (5.4)$$

Proof. According to Lemma 3.1, we have

$$y = \mp \prod_{j=1}^{2n+2} (\lambda - \lambda_j)^{\frac{1}{2}} \underset{\zeta \rightarrow 0}{=} \mp \zeta^{-n-1} (1 + \alpha_0 \zeta + O(\zeta^2)), \quad \text{as } P \rightarrow P_{\infty\pm}. \quad (5.5)$$

From (3.5), we obtain

$$\begin{aligned} F^{-1} &\underset{\zeta \rightarrow 0}{=} \zeta^{n+1} (F_{-1} + F_0 \zeta + O(\zeta^2))^{-1} \\ &\underset{\zeta \rightarrow 0}{=} \zeta^{n+1} (u^{-1} - u^{-2} F_0 \zeta + O(\zeta^2)), \quad \text{as } P \rightarrow P_{\infty\pm}, \end{aligned} \quad (5.6)$$

and

$$G \underset{\zeta \rightarrow 0}{=} -\frac{1}{2} w \zeta^{-n-1} (g_{-1} + g_0 \zeta + O(\zeta^2)), \quad \text{as } P \rightarrow P_{\infty\pm}. \quad (5.7)$$

Then according to the definition of ϕ in (5.3), we have

$$\begin{aligned} \phi &= \frac{y - G}{F} \\ &\underset{\zeta \rightarrow 0}{=} \left(\mp (1 + \alpha_0 \zeta + O(\zeta^2)) + \frac{1}{2} w (g_{-1} + g_0 \zeta + O(\zeta^2)) \right) (u^{-1} - u^{-2} F_0 \zeta + O(\zeta^2)) \\ &\underset{\zeta \rightarrow 0}{=} \begin{cases} -\frac{1+w}{u} + \frac{(1+w)u_x - uw_x}{2u^2}\zeta + O(\zeta^2), & \text{as } P \rightarrow P_{\infty+}, \\ \frac{1-w}{u} + \frac{(1-w)u_x + uw_x}{2u^2}\zeta + O(\zeta^2), & \text{as } P \rightarrow P_{\infty-}, \end{cases} \end{aligned} \quad (5.8)$$

which proves this lemma. \square

Hence the divisor of $\phi(P, x, t_m)$ is

$$(\phi(P, x, t_m)) = D_{\hat{\nu}_1(x, t_m), \dots, \hat{\nu}_{n+1}(x, t_m)} - D_{\hat{\mu}_1(x, t_m), \dots, \hat{\mu}_{n+1}(x, t_m)}. \quad (5.9)$$

Let $\omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P)$ denote the normalized Abelian differentials of the third kind holomorphic on $\mathcal{K}_n \setminus \{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)\}$ with simple poles at $\hat{\nu}_{n+1}(x, t_m)$ and $\hat{\mu}_{n+1}(x, t_m)$ with residues ± 1 , respectively, which can be expressed as

$$\begin{aligned} \omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) &= \left(\frac{y + G(\nu_{n+1}(x, t_m))}{\lambda - \nu_{n+1}(x, t_m)} - \frac{y - G(\mu_{n+1}(x, t_m))}{\lambda - \mu_{n+1}(x, t_m)} \right) \frac{d\lambda}{2y} \\ &\quad + \frac{\gamma_n}{y} \prod_{j=1}^{n-1} (\lambda - \gamma_j) d\lambda, \end{aligned} \quad (5.10)$$

where $\gamma_j \in \mathbb{C}$, $j = 1, \dots, n$, are constants that are determined by

$$\int_{\tilde{a}_j} \omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) = 0, \quad j = 1, \dots, n. \quad (5.11)$$

The explicit formula (5.10) then implies

$$\omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} (\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow \hat{\nu}_{n+1}(x, t_m), \zeta = \lambda - \nu_{n+1}(x, t_m) \\ (M(x, t_m) \pm \gamma_n)d\zeta, & \text{as } P \rightarrow P_{\infty \pm}, \zeta = \lambda^{-1} \\ (-\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow \hat{\mu}_{n+1}(x, t_m), \zeta = \lambda - \mu_{n+1}(x, t_m), \end{cases} \quad (5.12)$$

where $M(x, t_m) = \frac{1}{2}(\mu_{n+1}(x, t_m) - \nu_{n+1}(x, t_m))$. Therefore,

$$\int_{P_0}^P \omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \ln \zeta + \omega_0(\hat{\nu}_{n+1}(x, t_m)) + O(\zeta), & \text{as } P \rightarrow \hat{\nu}_{n+1}(x, t_m), \\ \omega_0^{\infty \pm} + (M(x, t_m) \pm \gamma_n)\zeta + O(\zeta^2), & \text{as } P \rightarrow P_{\infty \pm}, \\ -\ln \zeta + \omega_0(\hat{\mu}_{n+1}(x, t_m)) + O(\zeta), & \text{as } P \rightarrow \hat{\mu}_{n+1}(x, t_m), \end{cases} \quad (5.13)$$

where $\omega_0(\hat{\nu}_{n+1}(x, t_m))$, $\omega_0(\hat{\mu}_{n+1}(x, t_m))$, $\omega_0^{\infty \pm}$ are integration constants.

The Riemann theta function [3] is defined as

$$\theta(\underline{z}(P, D)) = \theta(\underline{K} - \mathcal{A}(P) + \mathcal{A}(D)), \quad (5.14)$$

where $P \in \mathcal{K}_n$, $D \in \text{Div}(\mathcal{K}_n)$, and $\underline{K} = (K_1, \dots, K_n)$ is defined by

$$K_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{\substack{k=1 \\ k \neq j}}^n \int_{\tilde{a}_k} \omega_k \int_{P_0}^P \omega_j, \quad j = 1, \dots, n. \quad (5.15)$$

Denote

$$\underline{\varrho}^{(l)} = \sum_{k=1}^n \int_{P_0}^{P_k^{(l)}} \underline{\omega} \quad (5.16)$$

with $P_k^{(1)} = \hat{\mu}_k(x, t_m)$, and $P_k^{(2)} = \hat{\nu}_k(x, t_m)$, $l = 1, 2$. Then we have

$$\theta(\underline{z}(P, D_{\hat{\mu}(x, t_m)})) = \theta(\underline{K} - \mathcal{A}(P) + \underline{\varrho}^{(1)}), \quad (5.17)$$

$$\theta(\underline{z}(P, D_{\hat{\nu}(x, t_m)})) = \theta(\underline{K} - \mathcal{A}(P) + \underline{\varrho}^{(2)}), \quad (5.18)$$

where $D_{\hat{\mu}(x, t_m)} = \sum_{j=1}^n \hat{\mu}_j(x, t_m)$, $D_{\hat{\nu}(x, t_m)} = \sum_{j=1}^n \hat{\nu}_j(x, t_m)$.

Theorem 5.1. Let $P = (\lambda, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$, $(x, t_m) \in \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected. Suppose $u(x, t_m), v(x, t_m), w(x, t_m) \in C^\infty(\Omega)$ satisfy the hierarchy of (2.11), and assume that λ_j , $1 \leq j \leq 2n+2$, in (3.13) satisfy $\lambda_j \in \mathbb{C} \setminus \{0\}$, and $\lambda_j \neq \lambda_k$ as $j \neq k$. Moreover, suppose that $D_{\hat{\mu}_1(x, t_m), \dots, \hat{\mu}_{n+1}(x, t_m)}$, or equivalently, $D_{\hat{\nu}_1(x, t_m), \dots, \hat{\nu}_{n+1}(x, t_m)}$ is nonspecial for $(x, t_m) \in \Omega$. Then u, w admit the following representation

$$\begin{aligned} w = & \left(\exp(\omega_0^{\infty+}) \theta(\underline{z}(P_{\infty+}, D_{\hat{\nu}(x, t_m)})) \theta(\underline{z}(P_{\infty-}, D_{\hat{\mu}(x, t_m)})) \right. \\ & \left. + \exp(\omega_0^{\infty-}) \theta(\underline{z}(P_{\infty+}, D_{\hat{\mu}(x, t_m)})) \theta(\underline{z}(P_{\infty-}, D_{\hat{\nu}(x, t_m)})) \right) \\ & \div \left(\exp(\omega_0^{\infty+}) \theta(\underline{z}(P_{\infty+}, D_{\hat{\nu}(x, t_m)})) \theta(\underline{z}(P_{\infty-}, D_{\hat{\mu}(x, t_m)})) \right. \\ & \left. - \exp(\omega_0^{\infty-}) \theta(\underline{z}(P_{\infty+}, D_{\hat{\mu}(x, t_m)})) \theta(\underline{z}(P_{\infty-}, D_{\hat{\nu}(x, t_m)})) \right), \end{aligned} \quad (5.19)$$

$$\begin{aligned} \frac{u_x}{u} + \frac{ww_x}{1-w^2} = & \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\hat{\nu}(x, t_m)}))} \\ & + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\hat{\mu}(x, t_m)}))} \\ & - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\hat{\mu}(x, t_m)}))} \\ & - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\hat{\nu}(x, t_m)}))} - 2\gamma_n. \end{aligned} \quad (5.20)$$

Proof. We introduce the local coordinate $\zeta = \lambda^{-1}$ near $P_{\infty\pm}$. From the definition (4.12) of the normalized bases ω_j , we have that

$$\begin{aligned}\underline{\omega} &= (\omega_1, \omega_2, \dots, \omega_n) = \mp \sum_{l=1}^n \underline{C}_l \frac{\lambda^{l-1} d\lambda}{\prod_{j=1}^{2n+2} (\lambda - \lambda_j)^{\frac{1}{2}}} \\ &= \pm \sum_{l=1}^n \underline{C}_l \zeta^{n-l} \prod_{j=1}^{2n+2} (1 - \lambda_j \zeta)^{-\frac{1}{2}} d\zeta \underset{\zeta \rightarrow 0}{=} \pm (\underline{C}_n + O(\zeta)) d\zeta, \text{ as } P \rightarrow P_{\infty\pm}.\end{aligned}\quad (5.21)$$

According to Riemann's vanishing theorem [15], the definition and asymptotic properties of ϕ , ϕ has expression of the following type

$$\phi(P, x, t_m) = N(x, t_m) \frac{\theta(\underline{z}(P, D_{\underline{\nu}(x, t_m)}))}{\theta(\underline{z}(P, D_{\underline{\mu}(x, t_m)}))} \exp \left(\int_{P_0}^P \omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) \right), \quad (5.22)$$

where $N(x, t_m)$ is independent of $P \in \mathcal{K}_n$. Given (5.13), we can derive that

$$\exp \left(\int_{P_0}^P \omega_{\hat{\nu}_{n+1}(x, t_m), \hat{\mu}_{n+1}(x, t_m)}^{(3)}(P) \right) \underset{\zeta \rightarrow 0}{=} \exp(w_0^{\infty\pm}) (1 + (M(x, t_m) \pm \gamma_n) \zeta + O(\zeta^2)), \text{ as } P \rightarrow P_{\infty\pm}.\quad (5.23)$$

Combining (5.17), (5.18) and (5.21), we obtain the following asymptotic expansion

$$\begin{aligned}\frac{\theta(\underline{z}(P, D_{\underline{\nu}(x, t_m)}))}{\theta(\underline{z}(P, D_{\underline{\mu}(x, t_m)}))} &= \frac{\theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \mathcal{A}(P_{\infty+}) - \mathcal{A}(P))}{\theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \mathcal{A}(P_{\infty+}) - \mathcal{A}(P))} \\ &\underset{\zeta \rightarrow 0}{=} \frac{\theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} - \underline{C}_n \zeta + O(\zeta^2))}{\theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} - \underline{C}_n \zeta + O(\zeta^2))} \\ &\underset{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\nu}(x, t_m)})) - \sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0} \zeta + O(\zeta^2)}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\mu}(x, t_m)})) - \sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0} \zeta + O(\zeta^2)} \\ &\underset{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\nu}(x, t_m)}))}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\mu}(x, t_m)}))} \left(1 - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\nu}(x, t_m)}))} \zeta + O(\zeta^2) \right) \\ &\quad \times \left(1 + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\mu}(x, t_m)}))} \zeta + O(\zeta^2) \right) \\ &\underset{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\nu}(x, t_m)}))}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\mu}(x, t_m)}))} \left[1 + \left(\frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\mu}(x, t_m)}))} \right. \right. \\ &\quad \left. \left. - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\nu}(x, t_m)}))} \right) \zeta + O(\zeta^2) \right], \text{ as } P \rightarrow P_{\infty+}.\end{aligned}\quad (5.24)$$

Substituting (5.23) and (5.24) into (5.22), we have

$$\begin{aligned} \phi(P, x, t_m) &= N(x, t_m) \exp(w_0^{\infty+}) \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}}(x, t_m)))} \\ &\times \left[1 + \left(M(x, t_m) + \gamma_n + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}}(x, t_m)))} \right. \right. \\ &\quad \left. \left. - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\nu}}}(x, t_m)))} \right) \zeta + O(\zeta^2) \right], \text{ as } P \rightarrow P_{\infty+}, \end{aligned} \quad (5.25)$$

Similarly, one can get

$$\begin{aligned} \phi(P, x, t_m) &= N(x, t_m) \exp(w_0^{\infty-}) \frac{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}}(x, t_m)))} \\ &\times \left[1 + \left(M(x, t_m) - \gamma_n + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\nu}}}(x, t_m)))} \right. \right. \\ &\quad \left. \left. - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}}(x, t_m)))} \right) \zeta + O(\zeta^2) \right], \text{ as } P \rightarrow P_{\infty-}, \end{aligned} \quad (5.26)$$

from (5.21)-(5.23). Comparing (5.25) and (5.26) with (5.4), we find

$$-\frac{1+w}{u} = N(x, t_m) \exp(w_0^{\infty+}) \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}}(x, t_m)))}, \quad (5.27)$$

$$\frac{1-w}{u} = N(x, t_m) \exp(w_0^{\infty-}) \frac{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}}(x, t_m)))}, \quad (5.28)$$

$$\begin{aligned} \frac{(1+w)u_x - uw_x}{2u^2} &= N(x, t_m) \exp(w_0^{\infty+}) \frac{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}}(x, t_m)))} \\ &\times \left(M(x, t_m) + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}}(x, t_m)))} \right. \\ &\quad \left. + \gamma_n - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\nu}}}(x, t_m)))} \right), \end{aligned} \quad (5.29)$$

$$\begin{aligned} \frac{(1-w)u_x + uw_x}{2u^2} &= N(x, t_m) \exp(w_0^{\infty-}) \frac{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\nu}}}(x, t_m)))}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}}(x, t_m)))} \\ &\times \left(M(x, t_m) + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\nu}}}(x, t_m)))} \right. \\ &\quad \left. - \gamma_n - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}}(x, t_m)))} \right). \end{aligned} \quad (5.30)$$

Eliminating from (5.27) and (5.28) the terms u and $N(x, t_m)$, we have (5.19). Substituting the RHS of (5.27) into (5.29), we find

$$\begin{aligned} -\frac{u_x}{2u} + \frac{w_x}{2(1+w)} = & M(x, t_m) + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}(x, t_m)}))} \\ & + \gamma_n - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty+}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty+}, D_{\underline{\hat{\mu}}(x, t_m)}))}. \end{aligned} \quad (5.31)$$

Similarly, we have

$$\begin{aligned} \frac{u_x}{2u} + \frac{w_x}{2(1-w)} = & M(x, t_m) + \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(2)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}(x, t_m)}))} \\ & - \gamma_n - \frac{\sum_{j=1}^n C_{jn} \partial_{\sigma_j} \theta(\underline{K} - \mathcal{A}(P_{\infty-}) + \underline{\varrho}^{(1)} + \underline{\sigma})|_{\underline{\sigma}=0}}{\theta(\underline{z}(P_{\infty-}, D_{\underline{\hat{\mu}}(x, t_m)}))}, \end{aligned} \quad (5.32)$$

from (5.28) and (5.30). Then eliminating from (5.31) and (5.32) the term $M(x, t_m)$ yields (5.20). \square

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